3.3: Complex Roots

CONSIDER $ay''+by'+cy=0 \Leftrightarrow ar^2+br+c=0$
WHERE
$$b^2-4ac<0$$

Ex $y''+y'+\frac{37}{4}y=0$

$$r^2+r+37/4=0$$

$$r=\frac{-1\pm\sqrt{1-4(1)(37/4)}}{2} \qquad i^2=-1$$

$$r=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-36}=-\frac{1}{2}\pm\frac{1}{2}6i$$

$$r_{1,2}=-\frac{1}{2}\pm 3i = \lambda \pm \omega i$$

$\lambda=-\frac{1}{2}$ = "REAL PART OF ROOT"
$\omega_{1,2}=\pm 3$ = "IMAGINARY PART OF ROOT"

WE WILL CONCLUDE

$$\left.\begin{aligned} y_1(t) &= e^{\lambda t}\cos(\omega t) \\ y_2(t) &= e^{\lambda t}\sin(\omega t) \end{aligned}\right\} \text{TWO INDEPENDENT SOL'N.}$$

GENERAL SOL'N:

$$\boxed{y=c_1e^{-\frac{1}{2}t}\cos(3t)+c_2e^{-\frac{1}{2}t}\sin(3t)}$$

ASIDE

Euler's Formula:

DEFINE
$$e^{\theta i} = \cos(\theta) + i\sin(\theta)$$

DO YOU KNOW TAYLOR SERIES? IF SO, WE CAN EXPLAIN THIS WITH TAYLOR SERIES

$$\cos(\theta) = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots$$

$$\sin(\theta) = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \dots$$

$$e^{\theta i} = 1 + (\theta i) + \frac{1}{2!}(\theta i)^2 + \frac{1}{3!}(\theta i)^3 + \frac{1}{4!}(\theta i)^4 + \dots$$

AND FINALLY NOTE $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$
$i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$,
and so on

$$\Rightarrow e^{\theta i} = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 \dots$$

$$\cos(\theta) + i\sin(\theta)$$

So
$$e^{(\lambda + wi)t} = e^{\lambda t} e^{wti}$$
$$= e^{\lambda t}(\cos(wt) + i\sin(wt))$$

$\Updownarrow$ SAME $\qquad \Updownarrow$ OPPOSITE

$$e^{(\lambda - wi)t} = e^{\lambda t}(\cos(-wt) + i\sin(-wt))$$

So
$$a_1 e^{(\lambda + wi)t} + a_2 e^{(\lambda - wi)t}$$
$$= \underbrace{(a_1 + a_2)}_{c_1} e^{\lambda t}\cos(wt) + \underbrace{(a_1 - a_2) i}_{c_2} e^{\lambda t}\sin(wt)$$

INITIAL CONDITIONS (A BIT MESSIER)

Ex) Solve $y'' + y' + \frac{37}{4} y = 0 \quad \begin{cases} y(0) = 2 \\ y'(0) = 8 \end{cases}$

STEP 1 $r_{1,2} = -\frac{1}{2} \pm 3i$ (FROM EARLIER)

$$y = c_1 e^{-\frac{1}{2}t} \cos(3t) + c_2 e^{-\frac{1}{2}t} \sin(3t)$$

$$y = e^{-\frac{1}{2}t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right)$$

STEP 2

$$y(0) = 2 \xRightarrow[\sin(0)=0]{\cos(0)=1} c_1 + 0 = 2$$

$$y' = -\frac{1}{2} e^{-\frac{1}{2}t}\left(c_1 \cos(3t) + c_2 \sin(3t)\right) + e^{-\frac{1}{2}t}\left(-3c_1 \sin(3t) + 3c_2 \cos(3t)\right)$$

$$y'(0) = 8 \Rightarrow -\frac{1}{2}(c_1 + 0) + (0 + 3c_2) \stackrel{?}{=} 8$$

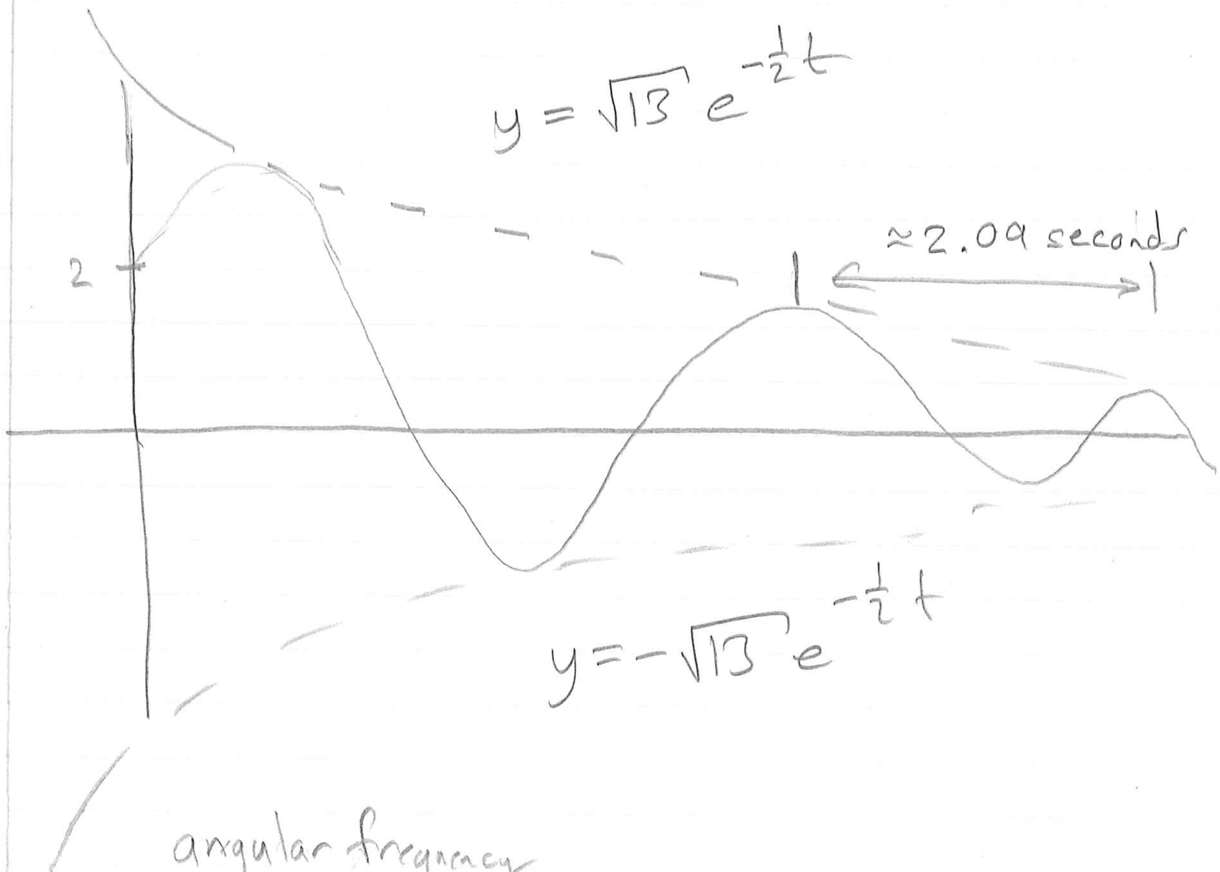
$$-\frac{1}{2} c_1 + 3c_2 = 8$$

$$-1 + 3c_2 = 8 \Rightarrow 3c_2 = 9$$

$$c_2 = 3$$

$$\boxed{y = e^{-\frac{1}{2}t}\left(2\cos(3t) + 3\sin(3t)\right)}$$

$$y = e^{-\frac{1}{2}t}\left(\underbrace{2\cos(3t) + 3\sin(3t)}\right)$$
$$= e^{-\frac{1}{2}t}\sqrt{2^2+3^2}\,\cos(3t-\delta)$$



angular frequency

$\omega = 2\pi f = 3 \ \frac{\text{RADIANS}}{\text{SECOND}}$ "CYCLES"

$f = \frac{3}{2\pi} \simeq 0.477 \ \frac{\text{WAVES}}{\text{SECOND}}$ ← frequency

$T = \frac{2\pi}{3} \simeq 2.094 \ \frac{\text{SECONDS}}{\text{WAVE}}$ ← WAVELENGTH

Ex) $y'' - 6y' + 10y = 0$ $\begin{cases} y(0) = -2 \\ y'(0) = 1 \end{cases}$

SOLVE! WHAT IS THE WAVELENGTH?

$$r^2 - 6r + 10 = 0 \Rightarrow r = \frac{6 \pm \sqrt{36 - 40}}{2}$$

$$r_{1,2} = -3 \pm \tfrac{1}{2}\sqrt{-4} = -3 \pm i \quad \begin{cases} \lambda = -3 \\ \omega = 1 \end{cases}$$

GENERAL FORM: $y = e^{3t}\left(c_1 \cos(t) + c_2 \sin(t)\right)$

INITIAL CONDITION:

$$y' = -3y(t) + e^{3t}\left(-c_1 \sin(t) + c_2 \cos(t)\right)$$

$$y(0) = -2 \Rightarrow c_1 + 0 \stackrel{?}{=} -2 \Rightarrow c_1 = -2$$

$$y'(0) = 1 \Rightarrow -3(c_1 + 0) + (0 + c_2) \stackrel{?}{=} 1 \Rightarrow -3c_1 + c_2 = 1$$

$$\Rightarrow -6 + c_2 = 1$$

$$c_2 = 7$$

$$y(t) = e^{3t}\left(-2\cos(t) + 7\sin(t)\right)$$

$$= e^{3t}\sqrt{4 + 49}\,\cos(t - \delta)$$

$$T = \frac{2\pi}{\omega} = 2\pi$$

$$= 6.283 \text{ seconds}$$